Quantum Error Correction Literature Survey

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***Abstract*—This document is intended to fulfill the requirements to complete a Literature Survey for Dr. Thoshitha Gamage's CS 454 Theory of Computation Class during the Spring 2023 semester (CRN 17991). The desired topic for this project is "Quantum Error Correction Automata and Grammars." The content herein presents a historical context of background theory, introduces quantum mechanics and computation principles, addresses recent advancements in the field, and concludes with a proposed hypothesis to be further researched and validated before the close of the semester.**

***Index Terms*—quantum, error correction, Hamming, Steane, introduction, automata, grammar, lattice, Turing, Deutsch, Feynman, complexity**

I. Introduction

Modern theory of computation can trace its roots back almost 100 years to Alonzo Church [1]. In 1936 and 1937, Church and Alan Turing independently arrived at what would now become known as the Church-Turing thesis which has formed the foundation of most aspects of classical computing [2], [3]. The abstract model known as a Turing machine is a mathematical concept upon which any classical algorithm has been known to be able to be implemented. Building upon concepts such as Finite and Push-Down Automata, the Turing machine concept has been exhaustively used in computer science when discussing computability, complexity, and efficiency.

Around the same time, great advancements in quantum physics were being made. In 1920's and 1930's, famous scientists such as Heinrich Heisenberg and Paul Dirac were building upon foundational research made in the late 1800's by Michael Faraday, Heinrich Hertz, Max Planck, and others. Questions about the very nature of reality as has been understood since the ancient Greeks was being called into question especially with the concept of wave-particle duality.

These two disciplines continued in parallel for a time, before physicist Richard Feynman changed everything with a 1982 paper asking if quantum physics can truly be simulated via classical computation [5]. What followed was an explosive growth in the field of theoretical quantum computing, and shortly thereafter came experimental physical quantum computers. The quantum computers of today are still in their infancy, and might be compared to classical computers in the near decades after Church and Turing's initial works. Just like with classical computers, quantum computers require error detection and correction to perform computations accurately. Including quantum error correction (QEC) into automata and grammars is an ongoing area of theoretical research, and is important as input into the practical design of experimental architecture for the decades ahead.

II. Outline

This literature survey is not meant to be an in-depth dive into quantum mechanics, but rather to provide chronological evidence of the published materials in the quantum computation field and where the research is at today. Sections III., IV., V., and VII. are quite linear in their chronology but section V.I. does overlap a bit in the timeline. This was done for readability and to illustrate that quantum computing and error correction is a wide field for both theorists and practical experimentation. Each field would not exist without the other, and they both have adapted over the past decades in parallel.

Because this writeup is intended to be the deliverable for Theory of Computation class, there are certain elements that are purposefully excluded to facilitate conciseness of the topic and to appropriately delineate deliverables in other classes. Simulating quantum systems, i.e. with GPU-enabled computations, is only mentioned in passing (see section). While there have been many advancements to simulating quantum systems via classical methods in recent years it is important to emphasize the limitations of these platforms to better appreciate what they can or cannot deliver.

III. Early Literature (1980's)

A. *Quantum Turing Machines*

In 1982, Richard Feynman's paper, "Simulating physics with computers", changed everything [5]. Feynman's argument was that physics could never be truly simulated with a traditional "universal" Turing machine! He goes into detail on how the physical world is quantum mechanical based on some of the physics literature at the time, and was not convinced that quantum physics could be simulated. Not being interested in arguing the point on approximate simulations of physical phenomena (this much was apparent) the question in mind was of exact (continuous) simulation. Citing a 1973 concept from Bennett, Feynman goes on to say how natural laws are all reversible, and therefore the simulation rules must be reversible as well [4]. This called into question concepts of simulating time and space with probabilities which was groundbreaking at the time. Exploring the topic of wave-particle duality, particle "spin" and photon polarization were determined to correlate quite nicely to "base" vs. "excited" states in a two-state system. But could quantum systems be probabilistically simulated by a classical computer? Feynman states that it is not possible due to what he describes as the

”hidden-variable” problem where it is impossible to represent results of quantum mechanics with a classical theoretical universal device (such as a Turing machine).

Shortly following Feynman is perhaps the second most important paper in. this topic, David Deutsch's ”Quantum theory, the Church-Turing principle and the universal quantum computer” [8]. Deutsch begins by picking apart the Church-Turing hypothesis by stating that it implies a ”physical assertion” in that ”every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means” [8]. The main argument against this implication was that it required classical physics and discrete variables for simulations where reality is quantum and continuous. Thus a desire for a universal quantum computer to be compatible with this notion of natural continuity emerged. This theoretical model would allow for ”quantum parallelism” in that certain probabilistic tasks can be performed faster by a quantum computer than a classical computer. The paper goes on to introduce Deutsch's version Benioff's quantum complexity theory and its comparisons against classical complexity theory [6].

Feynman did return to elaborate on his findings in light of Deutsch and others' work with an additional paper [10], but it is not discussed in this writeup for conciseness. Also of note is Chi-Chih Yao's work, ”Quantum circuit complexity” [11] of which is also not elaborated on in this writeup as Yao's concepts and foundations are strengthened and built upon later in the 1990's (discussed below).

B. *Reversible Logic*

Peres enters the fray with his contribution of constructing a quantum-mechanical Hamiltonian [9]. A Hamiltonian is a type of ”operator” that describes the total energy in a system. This reinforced important elementary concepts such as the law of conservation of energy. By devising a notation for the logical computational steps, Peres showed how each step in the computation must be locally reversible. According to the laws of nature, it was insufficient to show that global reversibility of the overall quantum operations was possible. These notions aligned with previous work regarding ”unitary” operations against individual components in the system [**?**]. This constraint may seem strange, but it becomes extremely important in the formation of quantum circuits and gate operations.

IV. Core Literature (1990's)

A. *Complexity*

Quantum mechanics is counterintuitive. Not only is it difficult to grasp conceptually, but by the early 1990's many scientists were wondering how practical a quantum computer even could be. In 1994 scientists were shocked by a paper from Peter Shor titled ”Algorithms for quantum computation: discrete logarithms and factoring” [12]. Shor first goes on to redefine complexity as a function of time and space. Considering how algorithms are generally considered efficient when the number of steps of the algorithm grows as some type of polynomial function against the size of the input (as compared to an exponential growth). The first half of this paper goes on to discuss the Bounded Probabilistic Polynomial complexity class and how it compares to the newly postulated Bounded Quantum Polynomial complexity class. The remaining portions of this paper go on to describe that by leveraging complexity concepts described earlier one could find discrete logarithms and factor integers on a quantum computer with a number of steps that is polynomial based on the input size. These were two well-studied problems at the time and reinvigorated desire into quantum computing research. Shor concludes this pivotal work by elaborating on strengths and weaknesses of this approach - namely the aforementioned unitary transformation limitations.

Formal evidence that quantum Turing machines violated the modern ”complexity theoretic” formation of the Church-Turing thesis began to arise [17]. The class BPP was shown to be contained in BQP. Building off of Deutsch [8], computer scientists began to view a quantum Turing machine (QTM) as a quantum physical analog of a probabilistic Turing machine. However, unlike probabilistic Turing machines, QTM allow ”branching” with complex ”probability amplitudes” that require time-reversible unitary type operations as mentioned above. Counter-evidence against Deutsch begins to arise stating that simulation time of a universal QTM can be accomplished in polynomial time and not exponential. Building off of Yao [11] it was elaborated on differences regarding QTM definitions, as the entire machine could be considered in a quantum superposition, or individual cells in the machine's tape could be considered in quantum superposition.

B. *Error Detection*

It was around the mid 1990's when operational scientists began to take a more serious look at the potential error in quantum systems. During this time some of the first steps were being taken to attempt to build a physical experimental system, and all sorts of questions began to arise. Quantum systems were extremely fragile, and susceptible to different types of noise and decoherence. Known for some time had been the idea of the ”no-cloning theorem” which stated that any quantum bit could not be ”observed” or measured without ”collapsing” it into its base or excited state - thereby losing its superposition [7]. Scientists were finding that the environmental particles surrounding the system attempting to be isolated were indeed ”measuring” the qubits in certain ways and by extension introducing unnecessary collapsing errors. It was beginning to look bleak for a physical realization of these technologies.

Following-up on his 1994 work, Peter Shor doubled down on his claimed practicality of quantum computing systems by introducing the first error correcting codes for quantum bits [14]. Even with the realization that quantum mechanics can speed up certain computations, the idea of decoherence destroys the information in the superposition of qubit states [14]. Shor makes an important distinction here regarding quantum Turing machines. By describing the contents of the memory cells in the tape as being able to be cast into the superposition of different states, the computer itself does not necessarily have to be in its own quantum state. The idea was that the computer performs deterministic unitary transformations on the quantum states, which paved the way to the modern quantum circuit model to replace the traditional quantum Turing machine model.

Previous works had all suggested that once a single qubit became entangled with the environment, then noise was introduced and the entire state of the system became corrupted. Shor argued this may be overly cautious. Building upon the classical analog of error correction, it was demonstrated how an arbitrary state of n qubits can be ”encoded” onto 9n physical qubits in a decoherence-resistant way [14]. This did indeed require an overhead of 8 additional ”ancilla” qubits to represent a single ”logical” qubit, but it was far less prone to error. The decoherence itself was measured without measuring the state of the qubits (therefore not violating the no-cloning theorem). But, an assumption was made: only one qubit was allowed to decohere and the others had to be unaffected. The paper concludes that decoherence itself must be a unitary operation following the aforementioned previous work.

Right alongside Shor in 1996 came Steane's very important paper, ”Error Correcting Codes in Quantum Theory” which revolutionized quantum error correction and is still widely used today [15]. In this paper, the Heisenberg uncertainty principle is revisited. This paper dove deep into theory and discussed projecting error correction codes in higher order Hilbert spaces to maintain data integrity. Considering a quantum system having a Hilbert space of 2^n dimensions (positive n). This could be a set of n two-state systems such as spin or excitation of particles. A general state of n particles can be written as a sum (entanglement) of product states. ”Words” in this context are product states, which is a unique string of bits. A ”code” is a set of words, all of the same length (number of bits). In this way, the Hamming distance is defined as the number of places/bits two words (of same length) differ. The minimum distance of a code is the smallest hamming distance allowed between between two code words. By combining this principle into the ”projection” into higher order Hilbert spaces, Steane was able to imrpove upon Shor's nine qubit error correction code by encoding a single logical qubit onto only five physical qubits! Great strides were being made to try and account for the noisy limitations of the existing experimental applications.

Unfortunately, more bad news soon followed with mention that the aforementioned encoding and decoding shemes are vulnerable to noise themselves [16]. Error correction is still argued to remain beneficial, but details and caveats begin to arise. Error correction steps regarding the encoding and decoding were not instantaneous and happened during an approximate finite time duration (albeit quite small). Any errors that occur during this time slice remained problematic. Storage and transmission of the data may have imperfect correction if affected during this time. For a given strength of environment interference, there is an optimal rate at which error correction should be performed. For large scale quantum calculations (i.e. very deep circuits) this can be a big challenge [19].

V. Advanced Theory (2000's)

A. *Quantum Automata and Grammars*

In the year 2000 Moore and Crutchfield published the work, ”Quantum automata and quantum grammars,” of which various topics such as Quantum Finite State (QFA) and Quantum Push-Down Automata (QPDA) are discussed. Included in this work is pummping lemmas, closure properties, Greibach normal forms, and more [20]. This work is particularly dense but aligns most appropriately with the Theory of Computation class. As such, this work is to be disected and scrutinized the most. As mentioned in the work's Introductory section, for the most part the quantum analog to the classical proofs are mostly copy and paste with a few notable exceptions. As such, the fine details of this work are not discussed in this literature survey, but the core concepts are to be expanded upon during the project final deliverable. Please note the conclusion section for next steps on this point.

B. *Fault-Tolerant Computation*

In the early to mid 2000's, more information and details on encoding logical qubits into higher Hilbert spaces with multiple physical qubits is described [19]. A key concept which has dramatic implications is the idea of ”error syndrome.” If a qubit experiences an error, it is able to encode its ”error syndrome” onto other qubits using tensor products. The syndrome itself is measured, thereby avoiding the collapse of the entire system. It is found to be possible to restore a state using only partial knowledge of the state in this way. A proof is given that it is not possible to correct arbitrary errors for a single logical qubit using four or less physical qubits (reasserting the five qubit minimum as stated by Steane [15]). When the measurement is performed against the syndrome (the ”error subspace”) it ”projects” the syndrome ”upwards” and corrects the error with its own type of unitary operation. In this way the ”entanglement was fought with entanglement” [26].

At this time quantum communication is beginning to separate itself somewhat from the umbrella study of quantum computation. While quantum communication and quantum computation are technically two different ”situations” that involve the manipulation of states by unitary operations where some information is eventually extracted, communication-specific topics involve multiple parties and focuses on transmission (potentially over noisy channels). Computation is usually just for one party and begins to narrow in on more complex unitary operations.

Error correction from Shor, Steane, and others were mostly surrounding case by case bases of errors. Eventually a more generalized model for quantum error correction emerges. In 2005, Kribs et. al produce a work that proposes ”operator quantum error correction” that attempts to unify different paradigms [24]. It sets up a standard error formula of which any error correction protocol can be built on top of. Upon reviewing this work, it is not entirely clear how accurate this finding is and requires further scrutiny.

VI. Experimental Results

A. Initial Physical Systems (1995-2006)

In 1995, some of the first experiments with physical quantum computers began to ermerge using Cold Trapped Ions [13]. These initial experiments only allowed for up to two qubits and were not much more than a proof of concept. The technology was still being decided on and figured out. Then in 1998 came the first experimental quantum error correction publication [18]! This latest system used Nuclear Magnetic Resonance (NMR) technology and demonstrated a degree of control over three spin-half particles had been successfully implemented, and it was able to do it at room temperature as compared to the absolute zero required by Cold Trapped Ions. The findings of these initial experiments concluded that the initial states of the ancillas for each encoding/decoding must be pure. A tradeoff was introduced to allow for "pseudo-pure" derivations which were less than ideal.

In the 1990's it was determined that fault-tolerant quantum computation was indeed possible, but Duplantier et. al showed in their 2005 work that suggests that error thresholds can be acceptable towards this effort [26]. Experimental results show that the accuracy in implementations of the quantum circuit model is on the order of only several percent in the best case, whereas most estimates of the threshold give percentage numbers of the order of 1×10^{-4} or less. Thresholds for Steane codes were also given.

B. Superconducting Qubits. (2007-present)

Sometime around the mid to late 2000's a new technology arrived and revolutionized the way physical quantum computers were built. Oliver and Welander give an overview of "superconducting" quantum bits in their 2013 publication [28]. This technology uses Cooper pairs of electrons in electronic circuits with Josephson tunnel junctions. When physical systems are built, they need to have special cooling so as not to disturb the system. They form "artificial" or "synthetic" atoms by utilizing electronic charge and analyzing across the Josephson junctions. These can be controlled via microwaves (much easier than the lasers used in the previous technologies) and reportedly maintain coherence times on the order of nanoseconds.

This coherence lifetime of nanoseconds has been recently improved upon with demonstrations exceeding 0.3 milliseconds [31]. By experimenting with different elements (namely moving away from niobium towards tantalum) scientists were able to achieve this milestone for a single qubit. A healthy dose of skepticism is necessary however, as nowhere in this publication is error correction mentioned, or of an entangled multi-qubit processor.

Superconducting qubits are the technology upon which most modern quantum computers are built today. Even though these systems use "synthetic" atoms for their computations, they are argued to be the same as pure atomic implementations, just a scaled up macro-scale version. The electromagnetic quantum physics behind Josephson junction is interesting in and of itself, and it gives theoretical physicists and computer scientists alike that someday a practical physical system may actually be realized.

VII. Experiment-Driven Theory (present)

A. Lattice Systems

Building upon the successes of superconducting qubits, Córcoles et. al make a significant stride towards fault-tolerant systems detailed in their 2015 publication, "Demonstration of a quantum error detection code using a square lattice of four superconducting qubits" [29]. It is argued here that by rearranging the circuit into a new type of two-by-two lattice structure (newly available under the superconducting technology) researchers were able to detect arbitrary error on an encoded two-qubit entangled state via "quantum non-demolition parity measurements" on another pair of syndrome qubits. Each qubit uses "quantum buses" where on a four qubit square lattice, diagonals are the data qubits and the corresponding counter-diagonals are the syndrome qubits. If no error is present, the syndrome qubits are both found to be in their ground state (as compared to their excited state). This work lays the foundation for larger lattice structures.

B. Qudits and Higher Dimensions

While quantum error correction is indeed the next immediate frontier and hurdle to surpass, other strides are being made in this area of research as well. Traditional qubits (logically encoded or physical) is a 2-dimensional system. The qubit is either excited or in its base state. Kues et. al go into detail on what could be coming next, namely d-dimensional systems [30]. This paper demonstrates on-chip generation of entangled quDit (not quBit) systems using photons. By entangling two qudits of dimension 10, instead of the traditional 2^n dimension they were able to obtain at least one hundred dimensions. More information on this experimentation is needed, and it is not a highly cited paper so some caution is to be exercised.

To be further analyzed include Low Density Parity Check (LDPC) codes, which are a type of linear error correction used when transmitting messages over a noisy channel. Construction of quantum LDPC (QLDPC) on N qubits has been a challenge for experimental and theoretical scientists alike [32].

VIII. Conclusion

Understanding the historical context as well as recent publications surrounding quantum error correction is important for this topic. With an understanding that at the conclusion of this course it is expected to give a presentation to the class, more formula and examples are to be provided instead of just going through the literature. Hagar's 2010 publication, "The Curse of the Open System," gives a very detailed introduction to quantum mechanics and computation and is easy to understand [**?**]. Lifting this material and translating and organizing it into has the potential to be a good part of the overall deliverable for this course. Duplantier et. al [26] have probably the best introduction into quantum errors, and the Unitary matrices used to detect and correct them. More information along the same vein of of this work is presented by Gottesman [**?**]. Devitt et. al [27] also have good beginner quantum error correction examples but with slightly different notation. This 2013 publication is also good because it preaches a healthy skepticism regarding large scale QEC still being out of reach with physical systems.

The intention is to further digest the information in the Theory of Computation class in light of material presented in Moore's and Crutchfield's work [20]. After providing a background on quantum computations, class concepts are to be combined with the topics presented here for a foundation laying of quantum automata and grammars. Building upon this foundation, new concepts regarding the introduction of error detection, correction, and fault tolerant concepts directly into the grammars themselves. Finally, once developed it is to be argued that these new grammars have direct application potentials against physical systems such as superconducting quantum computers.

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