Multiple Stochastic Integrals via Sequences in Geometric Algebras

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Submitted: April 11, 2005
MR Subject Classifications: 60B99, 60C05, 15A66, 15A75
keywords: algebraic probability, combinatorial probability, geometric algebra, Berezin integral

Abstract

A combinatorial construction of the multiple stochastic integral is developed using sequences in Clifford (geometric) algebras. In particular, sequences of Berezin integrals in an ascending chain of geometric algebras converge in mean to the iterated stochastic integral. By embedding such chains within an infinite-dimensional Clifford algebra, an infinite-dimensional analogue of the Berezin integral is discovered. Hermite and Poisson-Charlier polynomials are recovered as limits of Berezin integrals using this construction.

1 Introduction

While combinatorial approaches to multiple stochastic integrals are not new (cf. Rota and Wallstrom [6] and Anshelevich [1]), the use of sequences in geometric algebras to construct multiple stochastic integrals is original with the current author. As in the work of Rota and Wallstrom, the analysis underlying the geometric algebraic construction relies on the work of Engel [3].

All stochastic processes in the current work are assumed *a priori* to satisfy Engel's regularity conditions [3]. The multiple stochastic integral is then recovered as the limit in mean of sequences of Berezin integrals in an ascending chain of Clifford (geometric) algebras. This chain of Clifford algebras can be embedded within an infinite-dimensional Clifford algebra generated by the orthonormal basis of a separable Hilbert space. An infinite-dimensional analogue of the Berezin integral occurs as the limit in mean of a sequence of Berezin integrals, each of which considered with respect to a finite-dimensional subalgebra.

Definition 1.1. The L^2*-norm of* $X(\omega)$ is defined by

$$\|X(\omega)\| = \left(\mathbb{E}\left(|X(\omega)|^2\right)\right)^{\frac{1}{2}}. \tag{1.1}$$

A sequence of random variables $\{X_k(\omega)\}$ is said to *converge in mean* to $X(\omega)$ if

$$\mathbb{E}\left(|X_k(\omega) - X(\omega)|^2\right) \to 0 \text{ as } k \to \infty. \tag{1.2}$$

In this case, $X(\omega)$ is the *limit in mean* of the sequence $\{X_k(\omega)\}$, and one writes

$$X(\omega) = \underset{k\to\infty}{\mathrm{L.I.M.}}\, X_k(\omega).$$

By convention, given an interval $I = (s, t]$ and a stochastic process $X(t)$, the notation $X(I)$ denotes $X(t) - X(s)$.

Engel proved that given a system $\{X_1(t), \ldots, X_m(t)\}$ of $m \geq 1$ stochastic processes satisfying particular regularity conditions, one can write

$$\int \cdots \int_{0\leq t_1, t_2, \ldots, t_m \leq t} d\, X_1(t_1, \omega) \cdots d\, X_m(t_m, \omega) \tag{1.3}$$

as the limit in mean of sums of the form

$$\sum_{1\leq i_1, \ldots, i_m \leq q} X_1(I_{i_1}) X_2(I_{i_2}) \cdots X_m(I_{i_m}), \tag{1.4}$$

where $\{I_1, \ldots, I_q\}$ is some partition of $[0, t]$ into disjoint intervals.

Given a stochastic process $X(t, \omega)$, one can express the m^{th} iterated stochastic integral of $X(t, \omega)$:

$$X^{(m)}(t, \omega) = \int \cdots \int_{0\leq t_1, t_2, \ldots, t_m \leq t} d\, X(t_1, \omega) \cdots d\, X(t_m, \omega) \tag{1.5}$$

as the limit in mean of sums of the form

$$\sum_{1\leq i_1, \ldots, i_m \leq q} X(I_{i_1}) X(I_{i_2}) \cdots X(I_{i_m}). \tag{1.6}$$

Definition 1.2. Let $t > 0$, and for $n > 0$ fix a partition $\{0 < t_1 < t_2 < \cdots < t_n = t\}$ of $[0, t]$. Let $\mathcal{P}_n(t)$ denote the set $\{t_1, t_2, \ldots, t_n\}$, and let $X(t)$ be a stochastic process. Then the m^{th} *iterated stochastic integral* of $X(t)$ is given by

$$X^{(m)}(t) = \underset{n\to\infty}{\mathrm{L.I.M.}} \sum_{0 < t_{j_1}, t_{j_2}, \ldots, t_{j_{m-1}} < t} X((0, t_{j_1}]) X((t_{j_1}, t_{j_2}]) \cdots X((t_{j_{m-1}}, t]), \tag{1.7}$$

where the sum is taken over all $m - 1$-tuples in $\mathcal{P}_n(t)$.

Remark 1.3. The approach developed in this paper relies only on associativity of the stochastic processes. Any stochastic process whose multiple stochastic integral can be expressed by the limit in mean (1.7) can be recovered using the methods detailed here. This includes stochastic processes defined on noncommutative normed associative algebras.

2 Clifford (Geometric) Algebras

Combinatorial properties of the geometric product in the Clifford algebra $\mathcal{C}\ell_{n,n}$ are exploited to give an algebraic combinatorial construction of the multiple stochastic integral. The strategy is to define for each $n > 0$ a 2^n-dimensional associative algebra generated by commuting nilpotent elements. This algebra is constructed within a 2^{2n}-dimensional associative algebra generated by anti-commuting nilpotent elements. This algebra, in turn, is constructed within the 2^{4n}-dimensional Clifford (geometric) algebra $\mathcal{C}\ell_{2n,2n}$.

Definition 2.1. For fixed $n \geq 0$, let V be an n-dimensional vector space having orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The 2^n-dimensional *Clifford algebra* of signature (p, q), where $p + q = n$, is defined as the associative algebra generated by the collection $\{\mathbf{e}_i\}$ along with the scalar $\mathbf{e}_\emptyset = 1 \in \mathbb{R}$, subject to the following multiplication rules:

$$\mathbf{e}_i\,\mathbf{e}_j + \mathbf{e}_j\,\mathbf{e}_i = 0 \text{ for } i \neq j, \text{ and} \tag{2.1}$$

$$\mathbf{e}_i{}^2 = \begin{cases} 1, & \text{if } 1 \leq i \leq p \\ -1, & \text{if } p+1 \leq i \leq p+q = n. \end{cases} \tag{2.2}$$

We denote the Clifford algebra of signature (p, q) by $\mathcal{C}\ell_{p,q}$.

Generally the vectors generating the algebra do not have to be orthogonal. When they are orthogonal, as in the definition above, the resulting multivectors are called *blades*.

Let $[n] = \{1, , 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$ by underlined Roman characters. The basis elements of $\mathcal{C}\ell_{p,q}$ can then be indexed by these finite subsets by writing

$$\mathbf{e}_{\underline{i}} = \prod_{k \in \underline{i}} \mathbf{e}_k. \tag{2.3}$$

Arbitrary elements of $\mathcal{C}\ell_{p,q}$ have the form

$$u = \sum_{1 \leq i \leq \dim \mathcal{A}} u_{\underline{i}}\,\mathbf{e}_{\underline{i}}, \tag{2.4}$$

where $u_{\underline{i}} \in \mathbb{R}$ for each $1 \leq i \leq \dim \mathcal{A}$.

Definition 2.2. The *degree* of a monomial in $\mathcal{C}\ell_{p,q}$ is defined as the cardinality of its index. For example, $\deg \mathbf{e}_{1\,3\,4} = |\{1, 3, 4\}| = 3$.

Definition 2.3. For $0 \leq k \leq n$, the *degree-k part* of $u \in \mathcal{C}\ell_{p,q}$ is defined as the sum of degree-k monomials in the expansion of u. In other words,

$$\langle u \rangle_k = \sum_{\substack{\underline{i} \in 2^{[n]} \\ |\underline{i}| = k}} u_{\underline{i}}\,\mathbf{e}_{\underline{i}}. \tag{2.5}$$

Notation The notation $\langle\langle u\rangle\rangle_k$ is used to denote the sum of the coefficients in the degree-k part of u. That is,

$$\langle\langle u\rangle\rangle_k = \sum_{\substack{\underline{i}\in 2^{[n]}\\ |\underline{i}|=k}} u_{\underline{i}}. \tag{2.6}$$

Definition 2.4. The *Berezin integral* is the linear functional $\int_{\mathrm{B}} : \mathcal{C}\ell_{p,q} \to \mathbb{R}$ such that

$$\int_{\mathrm{B}} \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\, \mathbf{e}_{\underline{i}}\, d\mathbf{e}_1 \cdots d\mathbf{e}_n = u_{12\ldots n}. \tag{2.7}$$

In other words, the Berezin integral is the "top-form" coefficient in the expansion of u.

Remark 2.5. This use of the Berezin integral is not standard but follows naturally from Berezin's original construction on the Grassmann algebra [2].

Definition 2.6. Given arbitrary $u = \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\, \mathbf{e}_{\underline{i}}$ and $v = \sum_{\underline{i}\in 2^{[n]}} v_{\underline{i}}\, \mathbf{e}_{\underline{i}}$ the *Clifford inner product* of u and v is defined by

$$\langle u, v\rangle = \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\, v_{\underline{i}}. \tag{2.8}$$

Consequently, the expansion of $u \in \mathcal{C}\ell_{p,q}$ can be written

$$u = \sum_{\underline{i}\in 2^{[n]}} \langle u, \mathbf{e}_{\underline{i}}\rangle\, \mathbf{e}_{\underline{i}}. \tag{2.9}$$

This inner-product defines a norm on $\mathcal{C}\ell_{p,q}$ by

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}}. \tag{2.10}$$

This norm is referred to as the *Clifford inner-product norm.*

The reader is referred to works such as Lounesto [4] and Porteous [5] for essential background information on Clifford algebras.

Definition 2.7. For any $n > 0$, let $\mathcal{G}_n$ denote the associative algebra generated by the elements $g_i = \mathbf{e}_i + \mathbf{e}_{n+i} \in \mathcal{C}\ell_{n,n}$. Evidently, $\mathcal{G}_n$ is the associative algebra spanned by basis elements of the form

$$\begin{cases} \text{scalars: } g_0 = 1 \in \mathbb{R} \\ \text{vectors: } g_1, \ldots, g_n \\ \text{bivectors: } g_i\, g_j = g_{ij} \;\text{ where } 0 < i < j \leq n \\ \vdots \\ \text{n-vector: } g_1\, g_2 \,\cdots\, g_n \end{cases} \tag{2.11}$$

subject to the multiplication rules

$$\begin{cases} g_i\, g_j = -g_j\, g_i \\ g_1\, g_1 = g_1^2 = g_2^2 = \cdots = g_n^2 = 0. \end{cases} \tag{2.12}$$

As shorthand, denote the product $g_i\, g_j$ as g_{ij}. Further, allow $\underline{i}$ to represent a canonically ordered multi-index consisting of some subset of $[n] = \{1, 2, \ldots, n\}$, where it is assumed that $\underline{i} = 0$ corresponds to $\emptyset \in 2^{[n]}$. Thus arbitrary elements of $\mathcal{G}_n$ have the form

$$u = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\, g_{\underline{i}}\,, \tag{2.13}$$

where $u_{\underline{i}} \in \mathbb{R}$ for all $\underline{i} \in 2^{[n]}$ and $g_{\underline{i}} = \prod_{k \in \underline{i}} g_k$. The *degree* of a term $u_{\underline{i}}\, g_{\underline{i}}$ is defined as the cardinality of the index $\underline{i}$.

As before, the *Berezin integral* of $u \in \mathcal{G}_n$ is defined by

$$\int_{\mathrm{B}} u\, dg_1 \cdots dg_n = u_{[n]}. \tag{2.14}$$

Let $N = 2n$ and let $\mathbb{G} \subset \mathcal{G}_N$ be any collection of pairwise disjoint bivectors. In other words, $\mathbb{G}$ is a collection of bivectors $\{g_{ij}\}$ such that

$$g_{ij}, g_{k\ell} \in \mathbb{G} \Rightarrow \{i, j\} \cap \{k, \ell\} = \emptyset. \tag{2.15}$$

Clearly the maximal order of such a collection is $\frac{N}{2} = n$. Denote by $\mathbb{G}_{\max}$ the unique (up to isomorphism) collection of maximal order. Since the bivectors are disjoint, $\mathbb{G}_{\max}$ constitutes an abelian group.

Definition 2.8. Let $\mathcal{G}_n{}^{\mathrm{sym}}$ denote the associative algebra generated by the disjoint bivectors $\{\gamma_i\}_{1 \le i \le n} = \mathbb{G}_{\max}$ along with the scalar $\gamma_\emptyset = 1 \in \mathbb{R}$. Observe that

$$\gamma_i \gamma_j = \gamma_j \gamma_i, \text{ for } 1 \le i, j \le n, \text{and} \tag{2.16}$$

$$\gamma_i^2 = 0, \text{ for all } 1 \le i \le n. \tag{2.17}$$

As shorthand, denote the product $\gamma_i\, \gamma_j$ as γ_{ij}. Further, allow $\underline{i}$ to represent a canonically ordered multi-index consisting of some subset of $[n] = \{1, 2, \ldots, n\}$, where it is assumed that $\underline{i} = 0$ corresponds to $\emptyset \in 2^{[n]}$. Thus arbitrary elements of $\mathcal{G}_n{}^{\mathrm{sym}}$ have the form

$$u = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\, \gamma_{\underline{i}}\,, \tag{2.18}$$

where $u_{\underline{i}} \in \mathbb{R}$ for all $\underline{i} \in 2^{[n]}$ and $\gamma_{\underline{i}} = \prod_{k \in \underline{i}} \gamma_k$. The *degree* of a term $u_{\underline{i}}\, \gamma_{\underline{i}}$ is defined as the cardinality of the index $\underline{i}$.

As before, the *Berezin integral* of $u \in \mathcal{G}_n{}^{\mathrm{sym}}$ is defined by

$$\int_{\mathrm{B}} u \, d\gamma_1 \cdots d\gamma_n = u_{[n]}. \tag{2.19}$$

It should be clear that $\mathcal{G}_n{}^{\mathrm{sym}}$ is a 2^n-dimensional commutative subalgebra of the 2^{2n}-dimensional non-commutative algebra $\mathcal{G}_{2n}$, which in turn is a 2^{2n}-dimensional subalgebra of the 2^{4n}-dimensional Clifford algebra $\mathcal{C}\ell_{2n,2n}$. It should also be clear that arbitrary elements of $\mathcal{G}_n$ can be expanded as in (2.18) and that the definition of the Berezin integral is unchanged.

Remark 2.9. It is not difficult to see that the non-commutative algebra $\mathcal{G}_n$ is canonically isomorphic to Grassmann's exterior algebra.

3 Functions Defined on Partitions

In this section, functions are defined on the set of all partitions of $[n]$, denoted by $\mathcal{P}([n])$. A typical element $\pi \in \mathcal{P}([n])$ is a collection of disjoint subsets, called *blocks*, whose union is $[n]$, and $|\pi|$ will denote the number of blocks contained in π.

Let $f : 2^{[n]} \to \mathbb{R}$ be a function on the power set of $[n]$ with $f(\emptyset) = 1$. Define the function $h : \mathcal{P}([n]) \to \mathbb{R}$ by

$$h(\pi) = \prod_{b \in \pi} f(b). \tag{3.1}$$

Here each partition element $\pi \in \mathcal{P}([n])$ is assumed to be canonically ordered.

Since π is used to denote a partition of $[n]$, σ will be used to denote a permutation of the blocks in π. In other words, $\pi \in \mathcal{P}([n])$ and $\sigma \in \mathrm{S}_k$.

Theorem 3.1. *Let $0 < k \le n$. Then*

$$\int_{\mathrm{B}} \left(\sum_{\underline{i} \in 2^{[n]}} f(\underline{i}) \, \gamma_{\underline{i}} \right)^k d\gamma_1 \cdots d\gamma_n = \sum_{\substack{\pi \in \mathcal{P}([n]) \\ |\pi| = k}} \sum_{\sigma \in \mathrm{S}_k} h(\sigma(\pi)), \tag{3.2}$$

where S_k is the symmetric group on k elements; i.e., we sum over all permutations of blocks of each $\pi \in \mathcal{P}([n])$ such that $|\pi| = k$.

Proof. Begin with the following expansion:

$$\begin{aligned} \left(\sum_{\underline{i} \in 2^{[n]}} f(\underline{i}) \, \gamma_{\underline{i}} \right)^k &= \left(\sum_{\underline{i_1} \in 2^{[n]}} f(\underline{i_1}) \, \gamma_{\underline{i_1}} \right) \cdots \left(\sum_{\underline{i_k} \in 2^{[n]}} f(\underline{i_k}) \, \gamma_{\underline{i_k}} \right) \\ &= \sum_{\underline{i_1}, \ldots, \underline{i_k} \in 2^{[n]}} f(\underline{i_1}) \cdots f(\underline{i_k}) \, \gamma_{i_1} \cdots \gamma_{i_k}. \end{aligned} \tag{3.3}$$

Because ${\gamma_i}^2 = 0$ for $1 \le i \le n$, this sum is equal to

$$\sum_{\substack{\underline{i_1},\ldots,\underline{i_k} \in 2^{[n]} \\ \text{pairwise disjoint}}} f(\underline{i_1}) \cdots f(\underline{i_k})\, \gamma_{i_1} \cdots \gamma_{i_k}. \tag{3.4}$$

From this follows

$$\int\limits_{\mathrm{B}} \left(\sum_{\underline{i} \in 2^{[n]}} f(\underline{i})\, \gamma_{\underline{i}} \right)^k d\gamma_1 \cdots d\gamma_n = \sum_{\substack{\underline{i_1},\ldots,\underline{i_k} \in 2^{[n]} \\ \text{pairwise disjoint}}} f(\underline{i_1}) \cdots f(\underline{i_k})\, \gamma_{i_1} \cdots \gamma_{i_k} \Big]_{\underline{i_1} \cup \cdots \cup \underline{i_k} = [n]}. \tag{3.5}$$

I.e., the sum is restricted to those multi-indices whose disjoint union is all of $[n]$. Since the sum is over collections of k-blocks, the Berezin integral is the sum over all k-block partitions of $[n]$. Further, it is clear that the blocks recur in all possible permutations in the expansion. $\square$

Corollary 3.2. *Let $1 \le k \le n$. If f is commutative, then*

$$\int\limits_{\mathrm{B}} \left(\sum_{\underline{i} \in 2^{[n]}} f(\underline{i})\, \gamma_{\underline{i}} \right)^k d\gamma_1 \cdots d\gamma_n = k! \sum_{\substack{\pi \in \mathcal{P}([n]) \\ |\pi| = k}} h(\pi). \tag{3.6}$$

Further,

$$\int\limits_{\mathrm{B}} \exp \left(\sum_{\underline{i} \in 2^{[n]}} f(\underline{i})\, \gamma_{\underline{i}} \right) d\gamma_1 \cdots d\gamma_n = \sum_{\pi \in \mathcal{P}([n])} h(\pi). \tag{3.7}$$

Example 3.3. Fix $n > 1$, let $[n] = \{1, 2, \ldots, n\}$, and define

$$\eta_n = \sum_{\underline{i}} \gamma_{\underline{i}} \in {\mathcal{G}_n}^{\mathrm{sym}}. \tag{3.8}$$

Then for $1 \le k \le n$,

$$\int\limits_{\mathrm{B}} {\eta_n}^k \, d\gamma_1 \cdots d\gamma_n = k! \begin{Bmatrix} n \\ k \end{Bmatrix}, \tag{3.9}$$

where $\left\{ n \atop k \right\}$ denotes the *Stirling number of the second kind*, which represents the number of ways a set of n elements can be partitioned into k nonempty subsets.

Letting B_n denote the n^{th} Bell number, defined as the number of ways of partitioning a set of n elements into nonempty subsets, one further finds

$$\int\limits_{\mathrm{B}} e^{\eta_n} \, d\gamma_1 \cdots d\gamma_n = B_n. \tag{3.10}$$

4 The Evolution Sequence

Let $X(t,\omega)$ be any stochastic process whose multiple stochastic integral exists and is defined by (1.7). Let $t > 0$ be fixed and consider a sequence of partitions of the interval $(0,t]$ into n subintervals, $0 = t_0 < t_1 < \cdots < t_n = t$, such that the *mesh size* of these partitions goes to zero as $n \to \infty$. That is,

$$\lim_{n\to\infty} \max_{1\le k\le n} |t_k - t_{k-1}| = 0. \tag{4.1}$$

For each n, let these subintervals be labeled by $T_k = (t_{k-1}, t_k]$ for $1 \le k \le n$, and associate with each a bivector $\gamma_k \in {\mathcal{G}_n}^{\mathrm{sym}}$. I.e., $\gamma_k \sim (t_{k-1}, t_k]$. The set $\mathcal{I}_n(t) \subset 2^{[n]}$ of *admissible multi-indices* is defined by

$$\mathcal{I}_n(t) = \{\underline{k} \in 2^{[n]} : \bigcup_{\kappa\in\underline{k}} (t_{\kappa-1}, t_\kappa] = (t_\ell, t_r] \text{ for some } 0 \le t_\ell, t_r \le t\}. \tag{4.2}$$

Let $I_{\underline{i}}$ denote the subinterval $(t_\ell, t_k]$ associated with the bivector $\gamma_{\underline{i}} \in \mathcal{I}_n(t)$. In other words, each admissible multi-index is associated with a subinterval of $(0,t]$.

Define the notation $X((t_\ell, t_r], \omega) \equiv X(t_r,\omega) - X(t_\ell,\omega)$ and let the *evolution sequence* $\{\psi_n(X(t,\omega))\}_{n\ge1}$ *associated with the process* $X(t,\omega)$ be defined by

$$\psi_n(X(t,\omega)) = \sum_{\underline{i}\in\mathcal{I}_n(t)} X(I_{\underline{i}})\,\gamma_{\underline{i}}. \tag{4.3}$$

Theorem 4.1. *If $X(t,\omega)$ is a stochastic process whose m^{th} iterated stochastic integral exists, then*

$$\underset{n\to\infty}{\text{L.I.M.}} \left[\int_{\mathrm{B}} \psi_n\left(X(t,\omega)\right)^m \, d\gamma_1 \cdots d\gamma_n\right] = X^{(m)}(t,\omega), \tag{4.4}$$

where $X^{(m)}(t,\omega)$ is the iterated stochastic integral of $X(t,\omega)$.

Proof. Let $m > 0$ be fixed. For each $n > 0$, construction of the evolution sequence and Theorem 3.1 imply

$$\int_{\mathrm{B}} \psi_n(X(t,\omega))^m \, d\gamma_1 \cdots d\gamma_n = \sum_{\substack{0=t_0<t_1<\cdots<t_m=t \\ m\text{-subset partitions}}} \left(\sum_{\dot\pi\in \mathrm{S}_m} \prod_{j=1}^m X((t_{\dot\pi(j)-1}, t_{\dot\pi(j)}], \omega) \right). \tag{4.5}$$

Here the outer sum is taken over all m-interval partitions of $(0,t]$ having the n-interval partition of the evolution sequence as a common refinement.

By summing over all permutations of the set $\{i_1, \ldots, i_m\}$, one obtains

$$\int_{\mathrm{B}} \psi_n(X(t,\omega))^m \, d\gamma_1 \cdots d\gamma_n = \sum_{\substack{0=t_0<t_1,t_2,\ldots,t_{m-1}<t_m=t \\ m\text{-subset partitions}}} \left(\prod_{j=1}^m X((t_{j-1}, t_j], \omega) \right). \tag{4.6}$$

Hence, for each $n > 0$, a sum over sets in the product space $(0, t]^m$ with mesh size $\max_{1 \le k \le n} |t_k - t_{k-1}|$ is obtained. By hypothesis the multiple stochastic integral of $X(t, \omega)$ exists, so these sums converge in mean to a countably-additive stochastic measure on $(0, t]^m$ as the mesh size approaches zero, which happens as $n \to \infty$. □

5 Orthogonal Polynomials

Given a real-valued regular Poisson process $P(t, \omega)$, define the *compensated Poisson process* by

$$D(t, \omega) = P(t, \omega) - \mathbb{E}(P(t, \omega)). \tag{5.1}$$

This has mean zero and hence orthogonal increments.

Definition 5.1. For $m \in \mathbb{N}$, let

$$\mathrm{K}_m(u, t) = \frac{1}{m!} \sum_{q=0}^{m} \binom{m}{q} (-1)^q t^q u_{(m-q)}, \tag{5.2}$$

where $u_{(m-q)} = u(u-1)(u-2)\cdots(u-m+q-1)$. Then $\mathrm{K}_m(u, t)$ is the m^{th} Poisson-Charlier polynomial.

Definition 5.2. The n^{th} *generalized Hermite polynomial* is defined by

$$\mathrm{H}_n(u, t) = \frac{(-t)^n}{n!} e^{\frac{u^2}{2t}} \frac{\mathrm{d}^n}{\mathrm{du}^n} (e^{-\frac{u^2}{2t}}). \tag{5.3}$$

Theorem 5.3 (Engel)**.** *If $P(t, \omega)$ is the Poisson process and $D(t, \omega) = P(t, \omega) - t$, then*

$$\int \cdots \int_{0 \le t_1 < t_2 < \cdots < t_m \le t} dD(t_1, \omega) \cdots dD(t_m, \omega) = \mathrm{K}_m(P(t, \omega), t), \tag{5.4}$$

If $X(t, \omega)$ is standard Brownian motion, then

$$\int \cdots \int_{0 \le t_1 < t_2 < \cdots < t_m \le t} dX(t_1, \omega) \cdots dX(t_m, \omega) = \mathrm{H}_m(X(t, \omega), t), \tag{5.5}$$

where $\mathrm{H}_m(X(t, \omega), t)$ is the m^{th} Hermite polynomial.

These results are recovered using sequences of Berezin integrals.

Corollary 5.4. *Let $D(t, \omega)$ be the compensated Poisson process of* (5.1)*. For each $n \ge 1$, defining $\psi_n(D(t, \omega)) \in \mathcal{G}_n{}^{\mathrm{sym}}$ associated with $D(t, \omega)$, one finds*

$$\underset{n \to \infty}{\mathrm{L.I.M.}} \left[\int_{\mathrm{B}} \psi_n(D(t, \omega))^m \, d\gamma_1 \cdots d\gamma_n \right] = m! \, \mathrm{K}_m(P(t, \omega), t). \tag{5.6}$$

Corollary 5.5. *Let $X(t,\omega)$ be standard Brownian motion. For each $n \geq 1$, constructing $\psi_n(X(t,\omega)) \in \mathcal{G}_n{}^{\mathrm{sym}}$ associated with $X(t,\omega)$, one obtains*

$$\underset{n\to\infty}{\mathrm{L.I.M.}} \left[\int_{\mathrm{B}} \psi_n(X(t,\omega))^m \, d\gamma_1 \cdots d\gamma_n \right] = m! \, \mathrm{H}_m(X(t,\omega),t). \tag{5.7}$$

6 Berezin Integrals in Infinite-Dimensional Clifford Algebras

While the Berezin integral has no obvious extension to infinite-dimensional algebras, the discussions presented so far seem to suggest one such extension. Given an increasing sequence of Clifford algebras

$$\mathcal{C}\ell_{p_0,q_0} \subset \mathcal{C}\ell_{p_1,q_1} \subset \cdots \subset \mathcal{C}\ell_{p_n,q_n} \subset \cdots ,$$

where $p_i \leq p_j$, $q_i \leq q_j$, and $p_i + q_i < p_j + q_j$ for all $i < j$, one may wish to define a "Berezin integral" on sequences $\{x_i\}$, where $x_i \in \mathcal{C}\ell_{p_i,q_i}$ for each i.

Definition 6.1. Given a separable Hilbert space $\mathcal{H}$ having orthonormal basis $\{\mathbf{e}_i\}, (1 \leq i)$, define the *infinite-dimensional Clifford algebra* $\mathcal{C}\ell(\mathcal{H})$ as the associative $\mathbb{R}$-algebra generated by the vectors $\{\mathbf{e}_i\}$, satisfying

$$\mathbf{e}_i \, \mathbf{e}_j = -\mathbf{e}_j \, \mathbf{e}_i \, , \forall i \neq j \, , \tag{6.1}$$

and

$$\mathbf{e}_i{}^2 = \begin{cases} 1 \, , & \text{if } i \equiv 0 \pmod 2 \\ -1 \, , & \text{if } i \equiv 1 \pmod 2. \end{cases} \tag{6.2}$$

It is clear that $\mathcal{C}\ell_{p,q}$ can be embedded in $\mathcal{C}\ell(\mathcal{H})$ via the mapping $\iota : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell(\mathcal{H})$ defined by

$$\mathbf{e}_i \mapsto \begin{cases} \mathbf{e}_{2i} \, , & 1 \leq i \leq p \\ \mathbf{e}_{2(p-i)-1} \, , & p+1 \leq i \leq q. \end{cases} \tag{6.3}$$

Now the ascending chain is contained within an enveloping algebra:

$$\mathcal{C}\ell_{p_0,q_0} \subset \mathcal{C}\ell_{p_1,q_1} \subset \cdots \subset \mathcal{C}\ell_{p_n,q_n} \subset \cdots \subset \mathcal{C}\ell(\mathcal{H}) \, ,$$

where $p_i \leq p_j$, $q_i \leq q_j$, and $p_i + q_i < p_j + q_j$.

In particular,

$$\mathcal{C}\ell_{2,2} \subset \mathcal{C}\ell_{4,4} \subset \cdots \subset \mathcal{C}\ell_{2n,2n} \subset \cdots \subset \mathcal{C}\ell(\mathcal{H}) \, ,$$

and thus,

$$\mathcal{G}_1{}^{\mathrm{sym}} \subset \mathcal{G}_2{}^{\mathrm{sym}} \subset \cdots \subset \mathcal{G}_n{}^{\mathrm{sym}} \subset \cdots \subset \mathcal{C}\ell(\mathcal{H}).$$

Definition 6.2. Let $B_\infty : \ell^2(\mathcal{C}\ell(\mathcal{H})) \to \mathbb{R}$ denote the real-valued linear functional defined on the space of ℓ^2 sequences $\{u_n\}$ in $\mathcal{C}\ell(\mathcal{H})$ satisfying $u_n \in {\mathcal{G}_n}^{\mathrm{sym}}$ for each n by

$$B_\infty(\{u_n\}) = \lim_{n\to\infty} \int\limits_{\mathrm{B}} u_n \, d\gamma_1 \cdots d\gamma_n \,, \tag{6.4}$$

provided this limit exists, where $\int\limits_{\mathrm{B}} u_n \, d\gamma_1 \cdots d\gamma_n = u_{n[n]}$ is defined on the subalgebra ${\mathcal{G}_n}^{\mathrm{sym}}$ for each n.

It is not difficult to see that B_∞ is linear, homogeneous, and bounded. The extension to a linear mapping $\ell^2(\mathcal{A} \otimes \mathcal{C}\ell(\mathcal{H})) \to \mathcal{A}$ for any normed associative algebra $\mathcal{A}$ is also obvious.

The limit may be considered strong or weak, but in the context of the current work, one considers the limit in mean. Then

$$B_\infty\{\psi_n(X(t,\omega))^m\} = \underset{n\to\infty}{\mathrm{L.I.M.}} \left[\int\limits_{\mathrm{B}} \psi_n(X(t,\omega))^m \, d\gamma_1 \cdots d\gamma_n \right] = X^{(m)}(t,\omega). \tag{6.5}$$

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