1 Introduction

I hit a conceptual wall when I realized that the first problem I did where I thought I understood the weak form of an FEA problem was really just a Rayleigh-Ritz problem. This shouldn't be a big deal; if you can do Rayleigh-Ritz over the whole problem, doing FEA over smaller pieces (i.e. elements) shouldn't be that much different. But somehow it is causing a conceptual block for me.

So, I will now work through an axially loaded bar problem, most likely comparing FEA, Rayleigh-Ritz, and an analytical solution to prove to myself that I understand this stuff (mainly trying to get to an understanding of the weak form of an FEA problem).

2 Weak Formulation

I think the weak formulation for my problem (a uniform axial bar with one end built into a wall and a force applied to its free end) starts with the functional

$$\Pi_p = \sum_{i=1}^{N_{els}} \int \left(\frac{1}{2} \{\varepsilon\}^T [\mathbf{E}] \{\varepsilon\} \right) dV - \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{F}\} \, dV - \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{\Phi}\} \, dS - \{\mathbf{D}\}^T \{\mathbf{P}\} \tag{1}$$

which is equation 4.8-11 of Cook et.al. ([1]) with no initial stress or strain. The first summation of equation 1 represents the strain energy (I think that is the right term) of each element of the bar. The rest of the summations represent the work done by various kinds of forces acting on the bar. $\{\mathbf{F}\}$ refers to body forces (i.e. those acting on differential volumes of the element), $\{\mathbf{\Phi}\}$ refers to surface tractions (i.e. forces acting on a differential area of a surface). and $\{\mathbf{P}\}$ refers to concentrated loads acting at a point (a node) (see [1, pages 88-89]).

Ultimately, the functional will be transformed into

$$\Pi_p = \frac{1}{2} \{\mathbf{D}\}^T [\mathbf{K}] \{\mathbf{D}\} - \{\mathbf{D}\}^T \{\mathbf{R}\} \tag{2}$$

and the weak formulation will involve making the functional stationary by setting

$$d\Pi_p = [\mathbf{K}] \{\mathbf{D}\} - \{\mathbf{R}\} = 0 \tag{3}$$

and solving for $\{\mathbf{D}\}$.

2.1 From the Element Formulation to the Global Formulation

In order to prove to myself that I am understanding all of this, I will work through the transformation of the functional from the element-by-element formulation of equation 1 to the global formulation of equation 2. I will do this in general and for my specific problem of an axial bar under a load at the tip. This is just my re-hashing and interpretting [1, pages 159-161].

First, displacements within each element are interpolated from the nodal displacements for each element i

$$\{\mathbf{u}\} = [\mathbf{N}]\{\mathbf{d}\}_i \tag{4}$$

where $[\mathbf{N}]$ is the shape function matrix ([1, eq. 4.8-12]). For my axial bar problem, $\{\mathbf{u}\}$ will just be a scalar.

A relationship between strain and displacement is needed. In general,

$$\{\varepsilon\} = [\partial]\{\mathbf{u}\} \tag{5}$$

which leads to

$$\{\varepsilon\} = [\mathbf{B}]\{\mathbf{d}\} \tag{6}$$

(I think that should be $\{\mathbf{d}\}_i$) where

$$[\mathbf{B}] = [\partial][\mathbf{N}] \tag{7}$$

([1] has this as $\{\mathbf{N}\}$ in eq. 4.8-13, but I think that is a typo - $[\mathbf{N}]$ is not written as a vector anywhere else in this section.)

For my specific case,

$$\varepsilon = \frac{\partial u}{\partial x} \tag{8}$$

I think this means that

$$[\mathbf{B}] = \frac{\partial}{\partial x}[\mathbf{N}] \tag{9}$$

i.e.

$$\varepsilon = \frac{\partial}{\partial x}\left([\mathbf{N}]\{\mathbf{d}\}_i\right) \tag{10}$$

where $u = [\mathbf{N}]\{\mathbf{d}\}_i$ has been substituted into equation 8.

Since the $\{\mathbf{d}\}_i$ do not vary with x within the element,

$$\varepsilon = \frac{\partial}{\partial x}\left([\mathbf{N}]\right)\{\mathbf{d}\}_i \tag{11}$$

or

$$\varepsilon = [\mathbf{B}]\{\mathbf{d}\}_i \tag{12}$$

making use of (and more or less verifying) equation 9.

Substituting $\{\varepsilon\} = [\mathbf{B}]\{\mathbf{d}\}_i$ into equation 1 gives

$$\Pi_p = \sum_{i=1}^{N_{els}} \int \left(\frac{1}{2}\{\mathbf{d}\}_i^T [\mathbf{B}]^T [\mathbf{E}][\mathbf{B}]\{\mathbf{d}\}_i\right) dV - \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{F}\}\, dV - \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{\Phi}\}\, dS - \{\mathbf{D}\}^T \{\mathbf{P}\} \tag{13}$$

or

$$\Pi_p = \frac{1}{2}\sum_{i=1}^{N_{els}} \{\mathbf{d}\}_i^T [\mathbf{k}]_i \{\mathbf{d}\}_i - \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{F}\}\, dV - \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{\Phi}\}\, dS - \{\mathbf{D}\}^T \{\mathbf{P}\} \tag{14}$$

where

$$[\mathbf{k}]_i = \int [\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] \, dV \tag{15}$$

and again taking advantage of the fact that the $\{\mathbf{d}\}_i$ do not vary across the element, so they can be taken out of the integral of equation 13.

For my one dimensional case,

$$[\mathbf{k}]_i = \int E \left(\frac{\partial}{\partial x} [\mathbf{N}] \right)^T \left(\frac{\partial}{\partial x} [\mathbf{N}] \right) dV \tag{16}$$

Equation 14 can be further simplified by additional substitutions of $\{\mathbf{u}\} = [\mathbf{N}] \{\mathbf{d}\}_i$:

$$\Pi_p = \frac{1}{2} \sum_{i=1}^{N_{els}} \{\mathbf{d}\}_i^T [\mathbf{k}]_i \{\mathbf{d}\}_i - \sum_{i=1}^{N_{els}} \int \{\mathbf{d}\}_i^T [\mathbf{N}]^T \{\mathbf{F}\} \, dV - \sum_{i=1}^{N_{els}} \int \{\mathbf{d}\}_i^T [\mathbf{N}]^T \{\mathbf{\Phi}\} \, dS - \{\mathbf{D}\}^T \{\mathbf{P}\} \tag{17}$$

or

$$\Pi_p = \frac{1}{2} \sum_{i=1}^{N_{els}} \{\mathbf{d}\}_i^T [\mathbf{k}]_i \{\mathbf{d}\}_i - \sum_{i=1}^{N_{els}} \{\mathbf{d}\}_i^T \{\mathbf{r_e}\}_i - \{\mathbf{D}\}^T \{\mathbf{P}\} \tag{18}$$

where

$$\{\mathbf{r_e}\}_i = \int [\mathbf{N}]^T \{\mathbf{F}\} \, dV + \int [\mathbf{N}]^T \{\mathbf{\Phi}\} \, dS \tag{19}$$

Defining a matrix of ones and zeros for each element that selects the $\{\mathbf{d}\}_i$ from the global displacement vector $\{\mathbf{D}\}$ (i.e. a vector containing the displacements of all nodes)

$$\{\mathbf{d}\}_i = [\mathbf{L}]_i \{\mathbf{D}\} \tag{20}$$

and substituting this expression for $\{\mathbf{d}\}_i$ into equation 18 gives

$$\Pi_p = \frac{1}{2} \{\mathbf{D}\}^T \left(\sum_{i=1}^{N_{els}} [\mathbf{L}]_i^T [\mathbf{k}]_i [\mathbf{L}]_i \right) \{\mathbf{D}\} - \{\mathbf{D}\}^T \sum_{i=1}^{N_{els}} [\mathbf{L}]_i^T \{\mathbf{r_e}\}_i - \{\mathbf{D}\}^T \{\mathbf{P}\} \tag{21}$$

Defining a global stiffness matrix $[\mathbf{K}]$ and vector $\{\mathbf{R}\}$

$$[\mathbf{K}] = \sum_{i=1}^{N_{els}} [\mathbf{L}]_i^T [\mathbf{k}]_i [\mathbf{L}]_i \tag{22}$$

and

$$\{\mathbf{R}\} = \{\mathbf{P}\} + \sum_{i=1}^{N_{els}} [\mathbf{L}]_i^T \{\mathbf{r_e}\}_i \tag{23}$$

allows equation 21 to be rewritten as

$$\Pi_p = \frac{1}{2} \{\mathbf{D}\}^T [\mathbf{K}] \{\mathbf{D}\} - \{\mathbf{D}\}^T \{\mathbf{R}\} \tag{24}$$

which is the same as equation 2.

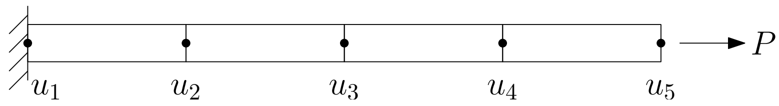

Figure 1: Axially loaded bar with 4 elements.

3 The Axially Loaded Bar

I think that applying all of this to the axially loaded bar problem starts with the shape functions

$$[\mathbf{N}] = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} \tag{25}$$

which leads to

$$[\mathbf{B}] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \tag{26}$$

according to equation 9. From this definition of $[\mathbf{B}]$, the element stiffness matrices can be found equation 15 where the integration over the element is preformed by substituting $dV = Adx$ and integrating from $x = 0$ to $x = L$:

$$[\mathbf{k}]_i = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \tag{27}$$

or

$$[\mathbf{k}]_i = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{28}$$

which agrees with [1, eq. 3.3-10], but has been derived from a functional.

If I model the axially loaded bar with 4 elements, as shown in Figure 1, the global displacement vector will be

$$[\mathbf{D}] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \tag{29}$$

and boundary condition will require that

$$u_1 = 0 \tag{30}$$

The $[\mathbf{L}]_i$ for each element wil be

$$[\mathbf{L}]_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

$$[\mathbf{L}]_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{32}$$

$$[\mathbf{L}]_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{33}$$

$$[\mathbf{L}]_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{34}$$

The contribution of each element to the global stiffness matrix will be

$$[\mathbf{K}]_1 = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 & 0 \\ -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

$$[\mathbf{K}]_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 \\ 0 & -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

$$[\mathbf{K}]_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{AE}{L} & -\frac{AE}{L} & 0 \\ 0 & 0 & -\frac{AE}{L} & \frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{37}$$

$$[\mathbf{K}]_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{AE}{L} & -\frac{AE}{L} \\ 0 & 0 & 0 & -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \tag{38}$$

so that the global stiffness matrix is

$$[\mathbf{K}] = \begin{bmatrix} \frac{A\,E}{L} & -\frac{A\,E}{L} & 0 & 0 & 0 \\ -\frac{A\,E}{L} & \frac{2\,A\,E}{L} & -\frac{A\,E}{L} & 0 & 0 \\ 0 & -\frac{A\,E}{L} & \frac{2\,A\,E}{L} & -\frac{A\,E}{L} & 0 \\ 0 & 0 & -\frac{A\,E}{L} & \frac{2\,A\,E}{L} & -\frac{A\,E}{L} \\ 0 & 0 & 0 & -\frac{A\,E}{L} & \frac{A\,E}{L} \end{bmatrix} \tag{39}$$

or

$$[\mathbf{K}] = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{40}$$

with the AE/L factored out.

For a point load P at the tip

$$[\mathbf{R}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{bmatrix} \tag{41}$$

and the problem is nearly ready to be solved for the nodal displacements $\{\mathbf{D}\}$ according to

$$[\mathbf{K}]\,\{\mathbf{D}\} = \{\mathbf{R}\} \tag{42}$$

once the boundary condition is properly handled.

But this seems like a problem, because the first row of this equation would seem to say that $u_1 + u_2 = 0$ which will be bad once the boundary condition of $u_1 = 0$ is plugged in.

[1, section 2.7] says that for any given row of equation 42, either $\{\mathbf{D}\}_j$ or $\{\mathbf{R}\}_j$ can be specified, but not both. That means that if I am specifying

$$\{\mathbf{D}\}_1 = 0 \tag{43}$$

$\{\mathbf{R}\}_1$ must be unknown, and therefore not equal to zero. But how does this play out in the weak formulation of the problem? The force at the left boundary of the bar does no work because $u_1 = 0$.

[1, section 2.7] suggests a procedure for handling this situation and finding the unknown $\{\mathbf{R}\}_1$. I think this will lead to the same solution as just eliminating the first row from equation 42 along with the first column of $[\mathbf{K}]$. I guess I will try this out.

$$\begin{bmatrix} -\frac{u_2\,A\,E}{L} \\ \frac{2\,u_2\,A\,E}{L} - \frac{u_3\,A\,E}{L} \\ -\frac{u_4\,A\,E}{L} + \frac{2\,u_3\,A\,E}{L} - \frac{u_2\,A\,E}{L} \\ -\frac{u_5\,A\,E}{L} + \frac{2\,u_4\,A\,E}{L} - \frac{u_3\,A\,E}{L} \\ \frac{u_5\,A\,E}{L} - \frac{u_4\,A\,E}{L} \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 0 \\ 0 \\ P \end{bmatrix} \tag{44}$$

Removing the first row and column of $[\mathbf{K}]$ gives

$$[\mathbf{K}'] = \begin{bmatrix} \frac{2\,A\,E}{L} & -\frac{A\,E}{L} & 0 & 0 \\ -\frac{A\,E}{L} & \frac{2\,A\,E}{L} & -\frac{A\,E}{L} & 0 \\ 0 & -\frac{A\,E}{L} & \frac{2\,A\,E}{L} & -\frac{A\,E}{L} \\ 0 & 0 & -\frac{A\,E}{L} & \frac{A\,E}{L} \end{bmatrix} \tag{45}$$

and the corresponding $\{\mathbf{D}\}$ and $\{\mathbf{R}\}$ vectors are

$$\{\mathbf{R}'\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ P \end{bmatrix} \tag{46}$$

$$\{\mathbf{D}'\} = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \tag{47}$$

Augmenting $\{\mathbf{R}'\}$ to $[\mathbf{K}']$ gives

$$[\mathbf{Aug}] = \begin{bmatrix} \frac{2\,A\,E}{L} & -\frac{A\,E}{L} & 0 & 0 & 0 \\ -\frac{A\,E}{L} & \frac{2\,A\,E}{L} & -\frac{A\,E}{L} & 0 & 0 \\ 0 & -\frac{A\,E}{L} & \frac{2\,A\,E}{L} & -\frac{A\,E}{L} & 0 \\ 0 & 0 & -\frac{A\,E}{L} & \frac{A\,E}{L} & P \end{bmatrix} \tag{48}$$

which can then be reduced to echelon form

$$[\mathbf{Sol}] = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & \frac{4\,L\,P}{A\,E} \end{bmatrix} \tag{49}$$

The bottom row can be solved for

$$u_5 = \frac{4\,L\,P}{A\,E} \tag{50}$$

And back substitution produces

$$u_4 = \frac{3\,L\,P}{A\,E} \tag{51}$$

$$u_3 = \frac{2\,L\,P}{A\,E} \tag{52}$$

$$u_2 = \frac{L\,P}{A\,E} \tag{53}$$

The total length of the bar is $l = 4L$, so these answers make conceptual sense.

$$u_5 = \frac{l\,P}{A\,E} \tag{54}$$

$$u_4 = \frac{3\,l\,P}{4\,A\,E} \tag{55}$$

$$u_3 = \frac{l\,P}{2\,A\,E} \tag{56}$$

$$u_2 = \frac{l\,P}{4\,A\,E} \tag{57}$$

And the force at the left boundary can be found from

$$R_1 = -\frac{u_2\,A\,E}{L} \tag{58}$$

or

$$R_1 = -P \tag{59}$$

which also makes sense.

4 Comparison to Analytical Solution

Having (apparently) solve the problem correctly using FEA and gotten an answer that makes physical sense, the next step will be comparing this answer to the analytical one and evaluating its merit/accuracy.

Based on the FEA solution, it seems like the analytical solution is

$$u(x) = \frac{x}{l}\frac{lP}{AE} = \frac{xP}{AE} \tag{60}$$

The bar is in a state of constant stress for all values of x:

$$\sigma_x = \frac{P}{A} = E\frac{\partial u}{\partial x} \tag{61}$$

so that

$$\frac{\partial u}{\partial x} = \frac{P}{AE} \tag{62}$$

Since u depends only on x,

$$\frac{du}{dx} = \frac{P}{AE} \tag{63}$$

and

$$du = \frac{P}{AE}dx \tag{64}$$

which means that

$$u = \frac{xP}{AE} + C \tag{65}$$

but the constant of intergration must vanish because $u(0) = 0$. So, the assumed solution based on FEA has been verified.

4.1 Rayleigh-Ritz Verifiacation

Even thought the solution has been verified already, further verification could come from using a Rayleigh-Ritz approach. The functional for the Rayleigh-Ritz problem is

$$\Pi_p = \int_0^l \frac{1}{2}E\frac{\partial u}{\partial x}^2 Adx - Pu(l) \tag{66}$$

The following expansion for $u(x)$ will be used

$$u(x) = a_2\, x^2 + a_1\, x \tag{67}$$

Substituting this experssion for $u(x)$ into equation 66 and integrating produces

$$\Pi_p = \frac{(4\,a_2^2\,l^3 + 6\,a_1\,a_2\,l^2 + 3\,a_1^2\,l)\ A\,E}{6} - \left(a_2\,l^2 + a_1\,l\right)\ P \tag{68}$$

Π_p must be stationary with respect to a_i, so the partial derivatives must vanish

$$\frac{\partial \Pi_p}{\partial a_1} = \frac{\left(6\, a_2\, l^2 + 6\, a_1\, l\right)\, A\, E}{6} - l\, P \tag{69}$$

$$\frac{\partial \Pi_p}{\partial a_2} = \frac{\left(8\, a_2\, l^3 + 6\, a_1\, l^2\right)\, A\, E}{6} - l^2\, P \tag{70}$$

These two equations can be solved for

$$\left[a_1 = \frac{P}{A\, E}, a_2 = 0\right] \tag{71}$$

which results in

$$u(x) = \frac{x\, P}{A\, E} \tag{72}$$

as expected, further verifying the solution.

References

[1] Cook, R. D., Malkus, D. S., Plesha, M. E., and Witt, R. J., 2002. *Concepts and Applications of Finite Element Analysis*. John Wiley & Sons.