Functional Analysis
Homework 4

Due Tuesday, 13 February 2018

Problem 1. A sequence \((e_n)\) of elements from a normed space \(X\) is said to be a Schauder basis if for every \(x \in X\), there is a unique sequence \((c_n)\) of scalars such that \(\sum_{i=1}^{n} c_n e_n \to x\). Show that if a normed space has a Schauder basis, then it is separable.

Problem 2. Let \(X\) be a vector space and \(Y\) subspace of \(X\). The quotient \(X/Y\) is the collection of cosets \(\{x + Y \mid x \in X\}\).

(a) Define an addition operation on \(X/Y\) by
\[
(x_1 + Y) + (x_2 + Y) := (x_1 + x_2) + Y.
\]
Prove that this addition on \(X/Y\) is well-defined.

(b) Define a scalar multiplication operation on \(X/Y\) in the natural way. That is,
\[
c(x + Y) := (cx) + Y.
\]
Prove that this scalar multiplication on \(X/Y\) is well-defined.

Problem 3. Suppose that \(X\) is a normed space and \(Y\) is subspace of \(X\).

(a) Prove that the function \(\|\cdot\|_0 : X/Y \to \mathbb{R}\) defined by
\[
\|x + Y\|_0 := \inf_{y \in Y} \|x + y\|
\]
is a pseudo-norm on \(X/Y\).

(b) Show that \(\|\cdot\|_0\) is a norm on \(X/Y\) if \(Y\) is closed.

(c) Prove that if \(X\) is a Banach space and \(Y\) is closed, then \(X/Y\) is a Banach space.

Problem 4. Give examples of subspaces of \(\ell^\infty\) and \(\ell^2\) which are not closed.

Problem 5. Consider the norms \(\|\cdot\|_1\) and \(\|\cdot\|_2\) on \(\mathbb{R}^n\). We already know that these are equivalent norms because \(\mathbb{R}^n\) is finite-dimensional. Prove the more explicit assertion that for all \(x \in \mathbb{R}^n\),
\[
\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1.
\]

Problem 6. Suppose \(X\) is a compact metric space and \(C \subseteq X\) is closed. Prove that \(C\) is compact.